Indian Statistical Institute
Mid-Semestral Examination 2003-2004
B.Math. (Hons.) III Year II Semester
Optimization

Time: 3 hrs Date:05-03-04

Answer all questions:

1. XYZ private enterprises manufacture three products, each of which utilizes some capacity in the three manufacturing departments, namely assembling, inspection and packing. Assuming all that is produced can be sold, find the production rate per week for each product so as to maximize the total profit. [Use Simplex Method].

Products	Per unit	Assembling	Inspection	Packing
B	10	.4	.12	.05
R	7	.6	.1	.03
G	6	.2	.16	.04
Available	Capacity per unit	1200 (hrs)	850	700

[4+8]

2. Use Bis M-method.
Maximize $v = 2x_1 + x_2 + 3x_3$
subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 5 \\ x_1 + 5x_2 + x_3 &\geq 0 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[10]

3. Let $S = \{x \colon Ax = b, x \geq 0\}$, where A is an $m \times n$ matrix of rank m, and be is an $m - vector$. Show that X is an extreme point of S if and only if A can be decomposed into $[B, N]$ such that

$$X = \begin{bmatrix} X_B \\ X_N \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$$

where B is an $m \times m$ invertible matrix satisfying $B^{-1}b \geq 0$. [8]

4. Use duality theorems to establish the following:
   Let A be an $m \times n$ matrix and c be an n-vector. Then exactly one of the following two systems has a solution.
   System 1 : $xX \leq 0$ and $c^t x > 0$ for some $x \in R^n$
   System 2: $A^t y = c$ and $y \geq 0$ for some $y \in R^m$
   Deduce
   For any $(m \times n)$ matrix A, exactly one of the following systems has a solution
   System 1: $Ax < 0$ for some $x \in R^n$
   System 2: $A^t y = 0, \quad y \geq 0$ for some nonzero $y \in R^m$.
   Hint: System 1 can be equivalently written as $Ax + se \leq 0$ for some $x \in R^n$, $s > 0$, $e = (1, 1, \ldots 1)$ vector of ones. [15]

5. Use the duality theorem to prove that the following linear programming problem is feasible but has no optimal solution.
   Minimizing $Z = 3x_1 - 5x_2 + x_3$
   subject to
$$x_1 - 2x_3 \geq 4$$
$$2x_1 - x_2 + x_3 \geq 2$$
$$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0.$$
   [5]

6. Let A be an $(n \times n)$ matrix. Obtain a set of sufficient conditions on c, b and A under which the following linear program is self dual:
   Min $(c^t x)$
   subject to
   $Ax \geq 0$
   $x \geq 0$ [5]

7. Consider the following linear programming problem and its optimal solution given in tableau 1.

   Minimize $Z = -2x, -4x_2 - 2.5x_3$

Subject to

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 &\le 60 \\ 2x_1 + x_2 + 2x_3 &\le 40 \\ x_1 + 3x_2 + 3x_3 &\le 30 \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

C_B	BasicVariable	x_1	x_2	x_3	x_4	x_5	x_6	Constant R.H.S
-2	x_1	1	0	$-\frac{6}{5}$	$\frac{3}{5}$	0	$-\frac{4}{5}$	12
0	x_5	0	0	3	-1	1	1	10
-4	x_2	0	1	$\frac{7}{5}$	$-\frac{1}{5}$	0	$\frac{9}{15}$	6
	$(C_j - Z_i)$	0	0	$\frac{7}{10}$	$\frac{2}{5}$	0	$\frac{4}{5}$	-Z=48

Write the dual of the problem and find the optimal values of the dual variables from the above tableau. [8]